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HYDRODYNAMIC FLOW AS CONGRUENCE OF GEODESIC LINES IN RIEMANNIAN SPACE-TIME

L. V. VEROZUB

Kharkov National University , 61077 Kharkov, Ukraine
Leonid.V.Verozub@univer.kharkov.ua

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It is shown that particles of perfect fluid in adiabatic processes move along geodesic lines of a Riemannian space-time.

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In papers [1,2] an interesting idea that fluid motion is equivalent to geodesic motion of fluid particles in non-Euclidean space-time has been considered. In the non-relativistic case it follows simply from the fact that for adiabatic processes in a perfect fluid with the density ρ and pressure P the first law of classical thermodynamics

$$d\Pi + Pd\left(\frac{1}{\rho}\right) = 0, \tag{1}$$

where $\rho\Pi$ is the fluid specific-energy density, leads to the equality

$$\frac{1}{\rho}\boldsymbol{\nabla}P = \boldsymbol{\nabla}\left(\Pi + \frac{P}{\rho}\right). \tag{2}$$

Therefore, along pathes of fluid particles Euler's equation in gravitational field

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\nabla}U - \frac{1}{\rho}\boldsymbol{\nabla}P, \tag{3}$$

where $d\mathbf{v}/dt = \partial\mathbf{v}/dt + \mathbf{v}\nabla$, takes the form

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\nabla}U_{eff} \tag{4}$$

where $U_{eff} = U - \left(\Pi + \frac{P}{\rho}\right)$. Such equations can be considered as non-relativistic limit of geodesic line of a Riemannian space-time of curvature other than zero.

However the proof and physical meaning of such fact in relativistic case still remains insufficiently clear.

In this paper we give a simple analysis of the problem and show that particles of perfect fluid in adiabatic processes indeed move along geodesic lines of a Riemannian space-time.

Let $d\sigma^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ be the metric differential form of Minkowski space-time E defined in some differentiable manifold $\mathcal{M}$, where $\eta_{\alpha\beta} = diag(-1,1,1,1)$ is the metric tensor in Cartesian coordinates. Let w be the enthalpy per unit volume and n be particles number density of a perfect fluid in E so that w/n is the enthalpy per particle. Consider also in $\mathcal{M}$ the second differential metric form

$$ds^2 = G_{\alpha\beta} dx^\alpha dx^\beta \tag{5}$$

where $G_{\alpha\beta} = \varkappa^2 \eta_{\alpha\beta}$,

$$\varkappa = \frac{w}{nmc^2} = 1 + \frac{\varepsilon}{mc^2} + \frac{P}{\rho c^2}, \tag{6}$$

ε is the fluid density energy, m is the mass of the fluid particles, and c is speed of light. The form (5) defines in $\mathcal{M}$ the structure of a Riemannian space-time V with curvature other than zero.

Let us show that the Lagrangian of the motion of particles of perfect fluid in adiabatic processes is of the form

$$L = -mc \left(G_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2} d\lambda \tag{7}$$

where λ is a parameter along 4-path of particles

First of all, it must be noted that the L describes the motion of the particles both in E and V . In the first case $G_{\alpha\beta}$ is some tensor field in E, in the second case it is a fundamental tensor of the Riemannian space-time V.

In Minkowski space-time E we can set the parameter $\lambda = \sigma$ which yields the following Lagrange equations:

$$\frac{d}{d\sigma} \left(\frac{G_{\alpha\beta} u^\beta}{(G_{\alpha\beta} u^\alpha u)^{1/2}} \right) - \frac{1}{2\,(G_{\alpha\beta} u^\alpha u)^{1/2}} \frac{\partial G_{\beta\gamma}}{\partial x^\alpha} u^\beta u^\gamma = 0 \tag{8}$$

where $u^\alpha = dx^\alpha/d\sigma$ is 4-velocity of the particles in E. In consequence of the equality $\eta_{\alpha\beta} u^\alpha u^\beta = 1$ these equations take a simple form

$$\frac{d}{d\sigma} (\varkappa u_\alpha) - \frac{\partial \varkappa}{\partial x^\alpha} = 0 \tag{9}$$

where $u_\alpha = \eta_{\alpha\beta} u^\beta$. For adiabatic processes [3]

$$\frac{\partial}{\partial x^\alpha} \left(\frac{w}{n} \right) = \frac{1}{n} \frac{\partial P}{\partial x^\alpha}, \tag{10}$$

and we arrive at the equations of the motion of particles in the form

$$w \frac{du_\alpha}{d\sigma} + u_\alpha u^\beta \frac{\partial P}{\partial x^\beta} - \frac{\partial P}{\partial x^\alpha} = 0. \tag{11}$$

where $du_\alpha/d\sigma = (\partial u_\alpha/\partial x^\epsilon)\, u^\epsilon$. It is the general accepted relativistic equations of the motion of fluid particles [3].

On the other hand, the Lagrange equations resulting from (7) are differential equations of a geodesic in V. If we set $\lambda = s$, the equations take the standard form of a congruence of geodesic lines :

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0, \tag{12}$$

where $du_\alpha/ds = \left(\partial u_\alpha/\partial x^\epsilon\right) u^\epsilon$ and

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} G^{\alpha\epsilon}\left(\frac{\partial G_{\epsilon\beta}}{\partial x^\gamma} + \frac{\partial G_{\epsilon\gamma}}{\partial x^\beta} - \frac{\partial G_{\beta\gamma}}{\partial x^\epsilon}\right) \tag{13}$$

$$= \frac{1}{\varkappa}\left(\frac{\partial \varkappa}{\partial x^\gamma}\delta^\alpha_\beta + \frac{\partial \varkappa}{\partial x^\beta}\delta^\alpha_\gamma - \eta^{\alpha\epsilon}\frac{\partial \varkappa}{\partial x^\epsilon}\eta_{\beta\gamma}\right). \tag{14}$$

The component

$$\Gamma^1_{00} = -\frac{1}{\rho c^2}\frac{\partial P}{\partial x^1} \tag{15}$$

and eqs. (12) lead to standard Euler equation in non-relativistic limit.

Thus, the motion of fluid particles in pseudo-Euclidean space-time are at the same time equations of the motion of these particles along geodesic lines of the Riemannian space-time V.

It is easily to expand the above result to the case of the motion of fluid in arbitrary Riemannian space-time with a metric tensor $g_{\alpha\beta}(x)$. In this case

$$G_{\alpha\beta} = \varkappa^2 g_{\alpha\beta}. \tag{16}$$

If the parameter $\lambda = \sigma$, the Lagrange equations (7) yield the equation which differ from (11) only by covariant derivatives instead the ordinary ones. These equations together with continuity equation are standard equations of fluid motion in General relativity. If we set $\lambda = s$, we will obtain these equations in the form of standard geodesic equations of the Riemannian space-time with the metric tensor $G_{\alpha\beta}$. More suitable to us as a parameter the coordinate time x^0 instead s so that the equations take the form

$$\ddot{x}^\alpha + \left(\Gamma^\alpha_{\beta\gamma} - c^{-1}\Gamma^0_{\beta\gamma}\dot{x}^\alpha\right)\dot{x}^\beta\dot{x}^\gamma = 0 \tag{17}$$

where $\dot{x}^\alpha = dx^\alpha/dt$. Zero component of these equations is satisfied identically, and rest equations are the ones for the 3-spacial velocity.

To make sure that these geodesic equations together with the continuity equations give the generally accepted description of fluid in General Relativity we consider the conditions of the fluid equilibrium in a given spherically-symmetric gravitational field. Since $\dot{x}^\alpha = \dot{x}^\alpha = 0$, the conditions of the equilibrium in spherical coordinates are

$$\Gamma^1_{00} = \frac{1}{2}G^{11}\left(G_{00}\right)' = 0 \tag{18}$$

where the prime denotes derivative with respect to the radial distance r. It means that $\left(\varkappa^2 g_{00}\right)' = 0$, or

$$\frac{\varkappa'}{\varkappa} = -\frac{\left(g_{00}\right)'}{2\, g_{00}}. \tag{19}$$

Due to the equality $\varkappa'/\varkappa = P'/\rho c^2 \varkappa$. and taking into account (6) we obtain for gas with $\varepsilon = 0$ the standard equation of the equilibrium in General Relativity [4]

$$\frac{(g_{00})'}{g_{00}} = -\frac{2P'}{\rho c^2 + P}. \tag{20}$$

Thus, the motion of an ideal fluid particles in a space-time with the metric tensor $g_{\alpha\beta}(x)$ at adiabatic processes take place along geodesic lines of the Riemannian space-time with the metric tensor (16).

This fact is of fundamental importance since it shows a bimetric nature of motion particles under the influence of force fields. Besides that, it gives sometime new insight into a description of hydrodynamic flow.

As a simple example consider a spherically-symmetric accretion a gas onto supermassive compact object in General Relativity. In the spherical coordinates (t, r, θ, φ) the Lagrangian (7) is of the form

$$L = -mc\,\varkappa^2 \left[A\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - c^2 C\right]^{1/2} \tag{21}$$

where $C = 1 - r_g/r$, $A = C^{-1}$, $r_g = 2\gamma M/c^2$, γ is the gravitational constant, M is the mass of the central object. Here $\varkappa$ is the function of the radial coordinate r, and the points denote the derivative with respect to time t. Owing to conservation of the energy E and angular momentum J of gas particles at the geodesic (free) motion in V, the equations of the motion in the plane $\theta = \pi/2$ takes the form

$$\dot{r}^2 = c^2 C^2 \left[1 - \varkappa^2 C/\overline{E}^2(1 + \overline{J}^2/\varkappa^2\overline{r}^2)\right] \tag{22}$$
$$\dot{\varphi} = cC\overline{J}r_g/\overline{r}^2\bar{E}$$

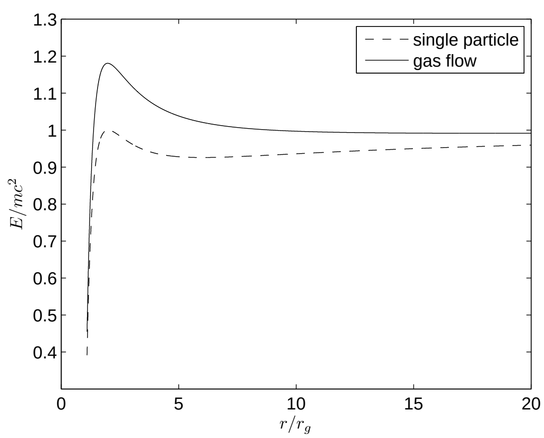

Fig. 1. An effective potential for the motion of particles of gas flow for Schwarzschild solution as compared with the efective potential of single particles. The particles energy $\overline{E} = 1$, the angular momentum $\overline{J} = 2$.

where $\overline{E} = E/mc$, $\overline{J} = J/r_g mc$. These equations gives more information about gas motion than Bernoulli equation, and allow to see easily the difference between motion of single particles and motion of a flow . In particular, if we setting in (22) $\dot{r} = 0$ we obtain the effective potential [5] for the motion of the flow particles, i.e. $\overline{E}^2$, as a function of $\overline{r}$:

$$E^2(\overline{r}) = \varkappa^2(1 - s1/\overline{r})(1 + \overline{J}^2/\varkappa^2\overline{r}^2).$$

Fig. 1 shows the effective potential for a single particle and for the relativistic gas flow where the function $\varkappa = 1/3\overline{r}$. It shows that the function $\varkappa(r)$ strongly influences on the postion of the minimum of the effective potential that defines the position of stable circular orbits of the gas around of the central object. The minimum of secular orbits of the gas lies at the distance $\overline{r} = 18.7$, while the one for the single particles lies at the distance $\overline{r} = 6$.

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